

An Economic Order Quantity Model for Weibull Deteriorating Items with Stock Dependent Consumption Rate and Shortages under Inflation

¹Bhaskar Bhaula, ²M.Rajendra Kumar

¹ Department of Mathematics, National Institute of Science and Technology
Berhampur, Odisha, India

² Department of Mathematics, National Institute of Science and Technology
Berhampur, Odisha, India

Abstract -This paper derives an optimal inventory replenishment policy for two parameters weibull deterioration with stock-dependent consumption rate, shortages under inflation and time discounting over a finite planning horizon. In a competitive super market, by and large customers are influenced by the quantity of stock displayed on the shelves. Hence for customers' inflow, this stock dependent demand factor is being included in our model paying more cost towards maintenance of high inventory. On the other hand shortage is a barrage for customers' inflow thus we have included a completely backlogged factor to avoid customers' dissatisfaction for long period. The convexity of the cost function has been proved analytically. The theoretical result reflected in this paper is also studied extensively through numerical example and sensitivity analysis.

Keywords - Inventory, inflation, Weibull deterioration, stock-dependent consumption rate and shortages

1. Introduction

In the super market, it is being observed that the demand of items go up and down according to the quantity of stock displayed. Lavin et al. [1972] has reported that large pile of goods displayed in supermarket has motivational effect on customers to buy more. Sufficient attention has been focused to the area that demands rate depends on the level of on-hand inventory during last two decades. The stock dependent consumption model was first developed by Gupta et al. [1986] and subsequently more researchers like Mandal et al. [1989], Datt et al. [1990], Chang et al. [1999], Teng et al [2002] have cultivated this area.

It is a natural phenomenon that each item in an inventory deteriorated spontaneously in course of time, certain

products like blood banks, food stuffs and medicines, chemicals, radioactive substances, etc are sufficiently deteriorating throughout their the normal storage period, so while developing an optimal inventory policy for such items; the loss of inventory because of deterioration cannot be ignored. It has been observed that the scientific community has continuously modified the deteriorating inventory models to become more practicable and realistic. It is being also observed that failure of many items can be articulated in terms of Weibull deterioration function. In last few years many workers have attempted to study inventory models for deteriorating systems. To get up to date research inspiration in this regard we referred articles of Ghare et al. [1963], Datt et al. [1991], Chung et al. [2007].

The inflation and time value of money were completely neglected in the past models discussed above. It is a common belief that the inflation and time value money has no momentous impact on any inventory policy. However, most of the developing countries have suffered a lot while purchasing oil, power, food, etc due to inflation thus in any inventory policy, the effect of inflation and time value money cannot be abandoned. The Pioneer investigator in this regard was Buzcote [1975] who developed an EOQ model with inflation under different types of pricing. Vrat et al. [1990] derived an inventory policy under constant inflation rate for initial stock-dependent consumption rate. In due course of time, Chung et al. [2001], Wee et al. [2001] and Kue-Lung Hou [2006] have studied the effects of inflation, time value of money and deterioration on inventory models. However, all the models discussed above have not included simultaneously stock-dependent consumption rate,

inflation and time value money under shortages over a finite planning horizon. The present manuscript has focused on mainly two parameters Weibull deteriorating items with stock-dependent consumption rate as well as shortages under inflation and time discounting over a finite planning horizon. The model is well discussed and solved analytically and the conditions for convexity of the total cost function has been established besides the study of the effect optimality applying sensitivity analysis vide numerical examples.

2. Notations and Assumptions

To develop a Mathematical model for the present article, the following notations and assumptions have been taken into deliberation:

r_1 = Inflation r

r_2 = Discount rate

$R = r_2 - r_1$ = Net discounted rate of inflation

T = Length of replenishment cycle

H = Length of finite planning horizon

$m = H / T$ = Number of replenishments during the entire planning horizon

$T_i = iT$ = Reorder times ($i = 0, 1, 2, 3 \dots m-1$)

$t_i = kiT$ = Times at which inventory level drops to zero where ($i = 1, 2, 3, \dots m$) $0 \leq k \leq 1$

$T_i - t_i$ = Time during which shortages occur ($i = 1, 2, 3, \dots m$)

I_m = Maximum inventory level

Q = Order quantity

$\lambda(t) = \alpha\beta t^{\beta-1}$ = Deterioration rate where $0 < \alpha < 1$, $\beta > 1$ and these are the parameters of Weibull function

$$B = \frac{1 - e^{-RH}}{1 - e^{-RH/m}}$$

A = Cost per replenishment

c = Cost of the item per unit

c_1 = Holding cost per unit per unit time

c_2 = Shortage cost per unit per unit time

I_0 = Initial inventory at $t = 0$

DCF = Discounted cash flow

❖ consumption rate $D(t) = a + bI(t)$, where $a \geq 0$, b is a stock-dependent consumption rate parameter, $0 \leq b \leq 1$ and $I(t)$ is the instantaneous inventory level

❖ the system worked for a prescribed period of a planning horizon

❖ the replenishment rate is infinite and lead time is zero

❖ variable rate of two parameters Weibull deterioration is applied to on-hand inventory only

❖ shortages are completely backordered

❖ product transaction are carried out by instantaneous cash flow

3. Mathematical Formulation

This article, assumes that some percentage of the on-hand inventory is lost due to deterioration and items are backordered when shortages reached at peak level. In the context of explaining the replenishment duration of the cycle of this paper 'T' is divided into two sub intervals $[0, t_1]$ and $[t_1, T]$. Throughout the interval $[0, t_1]$, the inventory level falls continuously due to combined effect of demand and deterioration until it becomes zero at $t = t_1$, consequently shortages are accumulated in the

interval $[t_1, T]$ till it reaches maximum I_s at $t = T$. At the end of each cycle, inventory reaches to maximum shortages, hence next order is placed to clear the backlog along with the inventory level to raise initial stage I_0 . We have divided the entire planning horizon H into m equal cycles of length $T = H / m$. As a result, the orders are placed at times $t_i = iT$ where $(i = 1, 2, 3, \dots, m-1)$.

To resolve the discussion mathematically the following differential equation is taken into contemplation.

$$\frac{dI(t)}{dt} + \alpha \beta t^{\beta-1} I(t) = -[a + bI(t)] \quad 0 \leq t \leq t_1 \quad (1)$$

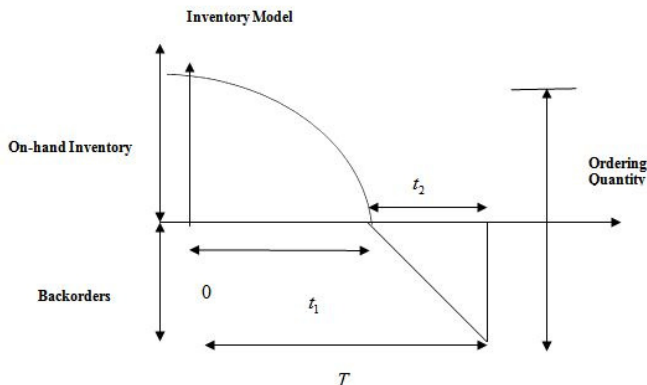
$$\frac{dI(t)}{dt} = -a \quad t_1 \leq t \leq T \quad (2)$$

The solutions of the above differential equations under boundary condition $I(t_1) = 0$ and with help of Taylor's expansion after neglecting higher powers of α and b are

$$I(t) = a \left[t_1 - t + \frac{\alpha(t_1^{\beta+1} - t^{\beta+1})}{\beta+1} + \frac{b(t_1^2 - t^2)}{2} \right]$$

$$0 \leq t \leq t_1 \quad (3)$$

$$I(t) = -a(t - t_1) \quad t_1 \leq t \leq T \quad (4)$$



The quantity of maximum initial inventory and maximum shortages can be obtained from equation (3) and (4) putting $t = 0$ and $t = T$ respectively

$$I_m(0) = a \left[t_1 + \frac{\alpha t_1^{\beta+1}}{\beta+1} + \frac{b t_1^2}{2} \right] \quad (5)$$

$$I_s(T) = -a(T - t_1) \quad (6)$$

At the commencement of each cycle, the order for replenishment of the stock is placed to clear the backlog and raise the inventory to initial level $I_m(0)$, hence for each cycle, Set up cost $C_r = A$ is to be well thought-out.

Inventory is purchased at the inauguration of each cycle to meet demand, deterioration and clear backlog, hence the amount paid towards purchasing of the material is

$$C_p = cI_m(0) + ce^{-RT} \int_0^{T-t_1} a dt = ca \left[t_1 + \frac{b t_1^2}{2} + \frac{\alpha t_1^{\beta+1}}{\beta+1} + (T - t_1)e^{-RT} \right] \quad (7)$$

Since the inventory is available during the interval $[0, t_1]$, hence the holding cost of the inventory is calculated for the time t_1 only. Using Taylor's series expansion and neglecting higher powers α and R , the total holding cost is

$$C_h = c_1 \int_0^{t_1} I(t) e^{-RT} dt = c_1 a \left[\frac{t_1^3}{6} - \frac{R t_1^4}{24} + \frac{\alpha \beta t_1^{\beta+2}}{(\beta+1)(\beta+2)} \right] \quad (8)$$

Due to stock-out situation, the shortages are happened and accumulated throughout the interval $[t_1, T]$ and reached

to the maximum level at $t = T$, hence the total shortage cost during the interval is

$$C_s = c_2 \int_{t_1}^T a(t-t_1)e^{-RT} dt = -\frac{ac_2 \left[\{1+R(T-t_1)\} e^{-RT} - e^{-Rt_1} \right]}{R^2} \quad (9)$$

The present value of total cost for the first cycle of the system is the sum up replenishment cost, purchasing cost, holding cost and shortage cost of the cycle, hence total present cycle cost

$$TRC = C_r + C_p + C_h + C_s \quad (10)$$

We have planned that there are m cycles throughout the planning horizon. Since the inventory is assumed to end at zero, an extra replenishment at $T_m = H$ is required to meet the backlog of the last cycle in the planning horizon. Hence there are $m+1$ replenishments during entire planning horizon. The first replenishment lot size is

$$I_m + \int_0^{T-t_1} a dt \text{ and last replenishment is } \int_0^{T-t_1} a dt. \text{ Hence}$$

the replenishment lot size for each cycle is

$$Q = I_m + a(T-t_1) \quad (11)$$

The total cost of the system throughout planning horizon H is

$$TC(m, k) = \sum_{j=0}^{m-1} TPC \cdot e^{-jRT} - Ae^{-RH} = TPC \left[\frac{1-e^{-RH}}{1-e^{-RH/m}} \right] - Ae^{-RH}$$

$$= B \left[A + ca \left\{ kT + \frac{b(kT)^2}{2} + \frac{\alpha(kT)^{\beta+1}}{\beta+1} + T(1-k)e^{-RT} \right\} + c_1 a \left\{ \frac{(kT)^2}{2} + \frac{\alpha\beta(kT)^{\beta+2}}{(\beta+1)(\beta+2)} - \frac{R(kT)^3}{6} + \frac{R^2(kT)^4}{24} \right\} + ac_2 \left\{ \frac{e^{-RkT} - (1+RT-RkT)e^{-RT}}{R^2} \right\} \right] - Ae^{-RH} \quad (12)$$

The present value of total system cost $TC(m, k)$ is a function of two variables m and k where discrete variable is m and k is a continuous variable. Hence for any given value of m the necessary condition for minimization can be obtained differentiating equation with respect to k

$$\frac{dTC(m, k)}{dk} = 0 \quad (13)$$

$$\Rightarrow \left[ca \left(T + b k T^2 + \alpha T^{\beta+1} k^{\beta} - T e^{-RT} \right) + \right.$$

$$c_1 a \left(k T^2 + \frac{\alpha \beta T^{\beta+2} k^{\beta+1}}{\beta+1} - \frac{R T^3 k^2}{2} + \frac{R^2 T^4 k^3}{6} \right) + \left. \frac{ac_2 T}{R} (e^{-RT} - e^{-RkT}) \right] = 0 \quad (14)$$

Sufficient condition for minimization can be obtained differentiating equation (14) with respect to k

$$\frac{d^2 TC(m, k)}{dk^2} = B \left[ca \left(b T^2 + \alpha \beta T^{\beta+1} k^{\beta-1} \right) + c_1 a \left(T^2 + \alpha \beta T^{\beta+2} k^{\beta} - R T^3 k + \frac{R^2 T^4 k^2}{2} \right) + ac_2 T^2 e^{-RkT} \right] \geq 0 \quad (15)$$

Since the second derivative is positive, hence the function is convex. Therefore it has minimum value. The optimum

value of k can be obtained from equation **Error! Reference source not found.** using Newton-Raphson's method for minimum possible positive integral value of m such that $TC(m, k)$ is minimum.

4. Numerical Examples

Numerical example with following data has been considered to study the effect of the optimality of the

$m=12$, $k=0.373764$, $\min TC(m, k)=15641.34$,
 $T=H/m=0.833333$, $t_1=kT=0.31147$ and $Q=501.7152$.

model through sensitivity. The values of inventory parameters

$\alpha = 0.05$, $\beta = 2.1$, $a = 600$ units/year,
 $b = 0.05$, $A = \$250$ / order, $c = 5$ / unit,

$c_1 = \$1.75$ /units/year, $R = 0.2$, $H = 10$ years.

From table-1, it is clear that the number of replenishments under which that the total cost of the system becomes minimum is 12. The optimization values of different terms associated with the model are

Sensitivity Analysis

TABLE 1: OPTIMUM SOLUTIONS UNDER SHORTAGES

m	k	$T = H / m$	$t_1 = kT$	$TC(m, k)$	Q
1	0.176394	10	1.76394	26414.47	6102.880
2	0.26003	5	1.30012	20761.34	3047.19
3	0.30036	3.3333	1.0012	18491.34	2024.74
4	0.323264	2.5	0.80816	17349.57	1514.797
5	0.33783	2	0.67566	16693.43	1209.717
6	0.34785	1.6667	0.57975	16287.66	1006.829
7	0.355145	1.428571	0.50735	16226.97	821.184
8	0.36068	1.25	0.45085	15857.78	753.868
9	0.365013	1.1111	0.40557	15749.84	669.723
10	0.36851	1	0.36851	15684.95	602.475
11	0.37136	0.90909	0.3376	15651.40	547.498
12	0.373764	0.83333	0.31147	156441.39	501.713
13	0.375778	0.769231	0.28906	15649.53	462.998
14	0.377524	0.714286	0.26966	15671.92	429.8288
15	0.379033	0.666667	0.25269	15705.73	401.094
16	0.380352	0.625	0.23772	15748.79	375.96
17	0.381514	0.588235	0.22442	15799.51	353.79
18	0.382554	0.555556	0.21253	15856.62	334.09
19	0.387477	0.526316	0.20183	15919.07	316.468
20	0.38434	0.5	0.19217	15986.07	300.612
21	0.385056	0.47619	0.18336	16056.99	283.268
22	0.38577	0.454545	0.17535	16131.29	273.232

TABLE 2: EFFECTS OF OPTIMALITY DUE TO THE CHANGE OF STOCK-DEPENDENT CONSUMPTION RATE b

b	k	$kT = t_1$	T	Q	$TC(m, k)$
0.05	0.37374	0.31145	0.83333	501.71499	15641.39
0.10	0.3555	0.29625	0.83333	502.8555	15680.35

0.15	0.339	0.2825	0.83333	503.78335	15715.71
0.20	0.32394	0.26995	0.83333	504.53919	15747.91

TABLE 3: EFFECTS OF OPTIMALITY DUE TO THE CHANGE OF NET DISCOUNT RATE OF INFLATION R

R	m	k	t_1	T	Q	$TC(m, k)$
0.05	13	0.533716	0.41055	0.76923	464.67889	28597.12
0.10	12	0.47916	0.3993	0.83333	502.9516	22973.81
0.15	12	0.426204	0.35517	0.83333	502.28112	18796.12
0.20	12	0.373704	0.31142	0.83333	501.7128	15643.63

TABLE 4: EFFECTS OF OPTIMALITY DUE TO THE CHANGE OF PURCHASING COST OF ITEM PER UNIT

c	m	k	$t_1 = kT$	T	Q	$TC(m, k)$
4.5	12	0.395258	0.32938	0.83333	501.93666	14361.18
5.0	12	0.373765	0.31147	0.83333	501.71524	15641.34
5.5	12	0.352561	0.2938	0.83333	501.51172	16361.44
6.0	12	0.331741	0.27645	0.83333	501.32597	18186.13

TABLE 5: EFFECTS OF OPTIMALITY DUE TO THE CHANGE OF HOLDING COST PER UNIT ITEM PER TIME

c_1	m	k	$t_1 = kT$	T	Q	$TC(m, k)$
1.50	12	0.3933	0.32775	0.83333	501.91587	15592.57
1.75	12	0.37376	0.31147	0.83333	501.715245	15641.34
2.00	12	0.35594	0.29662	0.83333	501.54322	15684.94
2.50	12	0.325020	0.27085	0.83333	501.26894	15759.60

TABLE 6: EFFECTS OF OPTIMALITY DUE TO THE CHANGE OF SHORTAGE COST PER UNIT ITEM PER UNIT TIME

c_2	m	k	$t_1 = kT$	T	Q	$TC(m, k)$
2.5	12	0.31036	0.25863	0.83333	501.1494	15418.80
3.0	12	0.37376	0.31147	0.83333	501.71524	15641.34
3.5	12	0.42606	0.35505	0.83333	502.28126	15825.94
4.0	12	0.4701	0.39175	0.83333	502.8316	15981.70

TABLE 7: EFFECT OF OPTIMALITY DUE TO THE CHANGE OF CONSUMPTION PARAMETER α

α	m	k	$t_1 = kT$	T	Q	$TC(m, k)$
500	12	0.37376	0.31147	0.83333	418.096	13028.811
600	12	0.37376	0.31147	0.83333	501.7152	15641.34
700	12	0.37376	0.31147	0.83333	585.3344	18253.86

TABLE 8: EFFECTS OF OPTIMALITY DUE TO THE CHANGE OF COST REPLENISHMENT PER ORDER

A	m	k	$t_1 = kT$	T	Q	$TC(m, k)$
200	12	0.37376	0.31147	0.83333	501.71524	15359.72
250	12	0.37376	0.31147	0.83333	501.71524	15641.34
300	12	0.37376	0.31147	0.83333	501.71524	15922.96

TABLE 9: EFFECT OF OPTIMALITY DUE TO CHANGE OF DETERIORATION PARAMETERS α AND β

α / β		1.9	2.0	2.1	2.2
0.05	k	0.372288	0.373044	0.37374	0.374364

	t_1	0.31024	0.31087	0.31145	0.31197
	T	0.833333	0.833333	0.833333	0.833333
	Q	506.12198	506.09864	506.08	506.064
	$TC(m,k)$	15644.05	15642.62	15641.39	15640.32
0.01	k	0.365436	0.3669	0.36822	0.369396
	t_1	0.30453	0.30575	0.30685	0.30783
	T	0.833333	0.833333	0.833333	0.833333
	Q	506.22219	506.18043	506.14611	506.11747
	$TC(m,k)$	15654.07	15651.31	15648.93	15646.89
0.15	k	0.359064	0.361164	0.363036	0.364752
	t_1	0.29922	0.30097	0.30253	0.30396
	T	0.833333	0.833333	0.833333	0.833333
	Q	506.30978	506.2526	506.20455	506.16575
	$TC(m,k)$	15663.55	15659.58	15656.15	15653.81
0.20	k	0.3531	0.35574	0.35814	0.36036
	t_1	0.29425	0.29645	0.29845	0.3003
	T	0.833333	0.833333	0.833333	0.833333
	Q	506.38619	506.3148	506.25599	506.20898
	$TC(m,k)$	15672.58	15667.48	15663.03	15659.22

The change in values of various parameters involved in any inventory system may occur due to uncertainties in decision making situations. Sensitivity analysis has vital role to study the effect of optimality due to the above changes. Depending upon the various results of our model reflected in the sensitivity analysis tables, the following inferences have drawn from.

❖ When the deterioration parameter α is increasing, both optimal cost and optimal ordering quantity are also rising but the deterioration parameter β has reverse affect of α .

❖ While the stock-dependent consumption rate b and consumption a are increasing, both optimal cost and ordering quantity are also rising moderately and significantly respectively.

❖ As soon as the net discount rate of inflation R is increasing the optimal cost is declining significantly.

❖ Whilst the unit purchasing cost C is increasing, the optima cost is highly escalating but ordering quantity decreasing slightly.

❖ Once the holding cost C_1 is increasing the optimal cost is also rising significantly but ordering quantity lessening slightly.

❖ When shortage cost C_2 is increasing, both optimal cost and optimal ordering quantity are also mounting significantly and slightly correspondingly.

❖ As the replenishment cost A is increasing, the optimal cost is also rising highly but this change has no effect on optimal ordering quantity.

❖ Shortage time is increasing on rising of α, b, R, c, c_1 but it is decreasing on increasing of C_2 whereas it is independent of a and A .

❖ In a nutshell, the optimal cost is escalating due to increasing of $\alpha, b, c, c_1, c_2, a$ and A but it is decreasing due to escalating of β and R .

5. Conclusions

An optimal inventory ordering policy has been developed considering stock-dependent consumption rate incorporating some realistic features such as two parameters Weibull deterioration, shortages fully backlogged, stock displayed on the shelves, inflation, time value of money and discounted cash flow approach. The quality and quantity of goods is declining in course of time due to deterioration, hence deliberation of Weibull distribution time varying deterioration function in place of constant deterioration taken in various models of this type carries a momentous meaning for perishable, volatile and failure of any kind of items. In a competitive super market, by and large customers are influenced by the quantity of stock displayed on the shelves. For customers' inflow, this stock dependent demand factor is being included in our model paying more cost towards maintenance of high inventory level.

The DCF approach permits a proper recognition of the financial implication of the opportunity cost in inventory analysis. Our inventory analysis table-4 indicates that the presence of inflation and time value of money reduces the inventory cost significantly low on rising of net discount rate of inflation hence this factor has positive impact on market. Shortage in inventory system is a natural phenomenon in real situations. Keeping all the above reality factors we have incorporated those in our model. Shortage is a barrage for customers' inflow thus we have included a completely backlogged factor to avoid customers' dissatisfaction for long period. The model is useful in the retail business community, dealing with fashionable clothes, household goods, electronic components and other products.

Thus the present model can further be enriched by incorporating delay in payment and progressive trade credit policy.

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First Author: Dr.Bhaskar Bhaula, M.Sc.(Mathematics)-1987,M.Phil.,-2006,Ph.D-2011.Worked as Associate Professor in the Dept. of Maths ,SMIT from 1989-2012.Presently working as Associate Professor in the Dept.of Mathematics,NIST.Number of papers published (7) and presented in National/International conferences. Permanent member of the professional bodies Orissa Mathematical Society and ISTE.

Second Author:Mr.M.Rajendra Kumar, M.Sc.(Mathematics)-1992,M.Phil.,-1999.Working as Associate Professor in the Dept. of Mathematics NIST since -2005.Participated in number o National/International conferences.